Rend. Istit. Mat. Univ. Trieste
Volume 49 (2017), 335–355
DOI: 10.13137/2464-8728/16219

A periodic problem for first order differential equations with locally coercive nonlinearities

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Dedicated to Professor Jean Mawhin on the occasion of his 75th birthday

Abstract. *In this paper we study the periodic boundary value problem associated with a first order ODE of the form $x' + g(t,x) = s$ where s is a real parameter and g is a continuous function, T-periodic in the variable t. We prove an Ambrosetti-Prodi type result in which the classical uniformity condition on $g(t,x)$ at infinity is considerably relaxed. The Carathéodory case is also discussed.*

Keywords: Periodic solutions, Multiplicity results, Local coercivity, Coincidence degree.
MS Classification 2010: 34B15, 34C25.

1. Introduction

This paper is concerned with the study of the periodic boundary value problem associated with the first order scalar ODE

$$(\mathscr{E}_s) \qquad x' + g(t,x) = s,$$

where s is a real parameter and g is a continuous function, T-periodic in the variable t.

Interest in this kind of parameter-dependent equations can be found in connection to the celebrated Ambrosetti-Prodi problem that was first investigated in the setting of the Dirichlet problem for elliptic PDEs (see [1, 2, 5]). The study of the Ambrosetti-Prodi problem for second order ODEs with periodic boundary conditions is a broad and active research area in which many investigators have been involved (see, for instance, [8, 23, 26] for some significant contributions in this field). In this latter context, the analysis is focused on the existence, nonexistence and multiplicity of (periodic) solutions for parameter dependent equations of the form

$$x'' + F(t,x,x') = s.$$

For the generalized Liénard equation

$$(\mathscr{LE}_s) \qquad x'' + f(x)x' + g(t,x) = s,$$

a relevant contribution in this direction is represented by the work of Fabry, Mawhin and Nkashama [8]. We recall here their result.

THEOREM 1.1. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, g is T-periodic in t and satisfies hypothesis*

$$(H) \qquad \lim_{|x|\to+\infty} g(t,x) = +\infty, \text{ uniformly in } t.$$

Then, there exists a number s_0 such that

1° *for $s < s_0$, equation $(\mathscr{LE}_s)$ has no T-periodic solutions;*

2° *for $s = s_0$, equation $(\mathscr{LE}_s)$ has at least one T-periodic solution;*

3° *for $s > s_0$, equation $(\mathscr{LE}_s)$ has at least two T-periodic solutions.*

The above theorem has motivated a rich area of research, including investigations on problems with singularities [9] and on nonlinear operators of p-Laplacian type [20].

The Ambrosetti-Prodi problem for first order periodic ODEs was studied by McKean and Scovel in [22] and by Vidossich in [29]. A version of Theorem 1.1 for equation $(\mathscr{E}_s)$ was carried out by Mawhin in [16, 17] and it can be stated as follows.

THEOREM 1.2. *Suppose that $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and T-periodic in t. Assume (H). Then, there exists a number s_0 such that*

1° *for $s < s_0$, equation $(\mathscr{E}_s)$ has no T-periodic solutions;*

2° *for $s = s_0$, equation $(\mathscr{E}_s)$ has at least one T-periodic solution;*

3° *for $s > s_0$, equation $(\mathscr{E}_s)$ has at least two T-periodic solutions.*

Notice that the results obtained for equation $(\mathscr{E}_s)$ can be stated also for

$$x' = q(t,x) \pm \theta,$$

where θ a real parameter. More precisely, we can reduce the above equation to $(\mathscr{E}_s)$, mainly in two ways. One is due to the obvious position $g(t,x) = -q(t,x)$ and $s = \pm\theta$. The other one follows from the change of variable $t \mapsto -t$, so that $g(t,x) = q(-t,x)$ and $s = \mp\theta$ (see also [16, Remark 1]).

As described in [18], a possible application of Theorem 1.2 is to the Riccati differential equation

$$x' + \gamma_2(t)x^2 + \gamma_1(t)x + \gamma_3(t) = 0.$$

In this case, the coercivity condition (H) is satisfied if

$$\gamma_2(t) \geq \kappa > 0, \quad \text{for all } t.$$

The motivation to study this topic is well presented in [18], by means of several interesting references describing the state of the art up to the middle of the Eighties.

REMARK 1.3. The works [16, 17, 18] of Mawhin, for equation $(\mathscr{E}_s)$, have stimulated a great deal of researches in this area. Even if, at first glance, the search of periodic solutions for equation $(\mathscr{E}_s)$ could appear “elementary”, it has been and, especially, it is still a source of interesting and, sometimes, challenging problems. Among the problems leading directly or indirectly to first order equations, we recall the study on the number of limit cycles for planar polynomial autonomous systems, which is connected to Hilbert sixteenth problem, and questions arising from single species population dynamics connected to periodic Kolmogorov equations (see the detailed presentations, as well as the comprehensive list of references, in [7, 25] that cover a great part of the literature concerning these equations up to the early 2000s).

In [28] we have proposed a possible variant of Theorem 1.1 for equation $(\mathscr{L}\mathscr{E}_s)$ in which the coercivity condition (H) is replaced by a local one, thus avoiding the uniformity in the variable t. Taking into account this generalization, the natural question which arises in the context of first order equations is whether the same outcome holds in the setting of Theorem 1.2. A clue that this conjecture is true can be found in the study of the Kolmogorov equation $x' = xh(t,x)$ and in the particular case of the Verhulst (logistic) equation, namely for $h(t,x) = r(t) - \gamma_2(t)x$. Indeed, from [3, 27, 31, 32], a classical result for equation

$$x' + \gamma_2(t)x^2 - r(t)x = 0,$$

with $r, \gamma_2 : \mathbb{R} \to \mathbb{R}$ continuous and T-periodic functions, is the existence of exactly two T-periodic solutions, the trivial one and another one positive, provided that

$$\int_0^T r(t)\,dt > 0 \quad \text{and} \quad \gamma_2(t) \geq 0 \;\forall\, t, \;\; \gamma_2 \not\equiv 0.$$

In the present paper we propose an extension of Theorem 1.2, in the spirit of [28]. In particular, we replace condition (H) by an average-type assumption at infinity of Gaetano Villari's type, which reads as follows.

(GV) *Given* $K_1 > 0$ *and* $K_2 > 0$, *for each* σ *there exists* $d_\sigma > 0$ *such that* $\frac{1}{T}\int_0^T g(t,x(t))\,dt > \sigma$ *for each* $x \in C_T$ *with* $|x(t)| \geq d_\sigma$ *for all* $t \in [0,T]$ *and such that* $|x|_{\max} \leq K_1|x|_{\min} + K_2$.

We remark also that an immediate consequence of condition (H) is that the function g is bounded from below. In our case, such lower bound is no more guaranteed and therefore we impose the following one-sided growth assumption:

(G_0) $\exists\, a_0, b_0 \in L^1([0,T],\mathbb{R}^+): \; g(t,x) \geq -a_0(t)|x| - b_0(t), \; \forall\, x \in \mathbb{R}, \; t \in [0,T].$

In this setting, we are in position to present our main result.

THEOREM 1.4. *Suppose that $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and T-periodic in t. Assume (G_0) and (GV). Then, there exists a number s_0 such that*

1° *for $s < s_0$, equation $(\mathscr{E}_s)$ has no T-periodic solutions;*

2° *for $s = s_0$, equation $(\mathscr{E}_s)$ has at least one T-periodic solution;*

3° *for $s > s_0$, equation $(\mathscr{E}_s)$ has at least two T-periodic solutions.*

A possible corollary of Theorem 1.4 is the following.

COROLLARY 1.5. *Let $\gamma_0, \gamma_1, \gamma_p : \mathbb{R} \to \mathbb{R}$ be continuous and T-periodic functions and let $p > 1$. Suppose that $\gamma_p(t) \geq 0$ for all t with $\gamma_p \not\equiv 0$. Then, for equation*

$$(\mathscr{R}\mathscr{E}_s) \qquad x' + \gamma_p(t)|x|^p + \gamma_1(t)x + \gamma_0(t) = s,$$

the following result holds. There exists a number s_0 such that:

1° *for $s < s_0$, equation $(\mathscr{R}\mathscr{E}_s)$ has no T-periodic solutions;*

2° *for $s = s_0$, equation $(\mathscr{R}\mathscr{E}_s)$ has at least one T-periodic solution;*

3° *for $s > s_0$, equation $(\mathscr{R}\mathscr{E}_s)$ has at least two T-periodic solutions.*

Looking again at the uniform condition (H) and applying it to $(\mathscr{R}\mathscr{E}_s)$, then we need to require that $\gamma_p(t)$ is positive and uniformly bounded away from zero. On the other hand, by Corollary 1.5, we observe that the coercivity condition in our setting is of local type. Notice also that $g(t,x) = \gamma_p(t)|x|^p + \gamma_1(t)x + \gamma_0(t)$ is not necessarily bounded from below but it satisfies the growth assumption (G_0).

The scheme of the proof already developed in [8, 16, 17] is reconsidered here to prove Theorem 1.4. In more detail, we combine topological degree arguments and the method of upper-lower solutions with some new tools adapted from [28]. We will also take advantage of some preliminary lemmas needed to treat the case of first order equations. We stress the fact that all our results will be formulated in the Carathéodory setting. In this manner we also improve some previous results in [24].

2. Preliminaries

In this section we deal with the periodic boundary value problem associated with the first order differential equation

$$x' + \psi(t, x) = 0, \tag{1}$$

where we assume that $\psi : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. As usual, by a T-periodic solution of (1) we mean a generalized solution $x : [0, T] \to \mathbb{R}$ of the equation (1) which satisfies the boundary condition

$$x(0) = x(T).$$

Equivalently, one could extend the map $\psi(\cdot, x)$ on $\mathbb{R}$ by T-periodicity and then consider T-periodic solutions $x : \mathbb{R} \to \mathbb{R}$ with x absolutely continuous (AC). In the frame of Mawhin's coincidence degree theory we will find a priori bounds and will provide existence results for periodic solutions of equation (1).

The standard setting to enter in the framework of the coincidence degree is the following. Let

$$X = C_T := \{x \in C([0, T]) : x(0) = x(T)\},$$

endowed with the norm $\|x\|_X := \|x\|_\infty$ and $Z = L^1([0, T])$ with the norm $\|x\|_Z := \|x\|_1$. Let $L : X \supseteq \mathrm{dom}L \to Z$ be defined as $Lx := -x'$, with

$$\mathrm{dom}L = W^{1,1}_T := \{x \in X : x \in AC\}.$$

According to [14], a natural choice for the projections is given by

$$Qx := \frac{1}{T} \int_0^T x(t)\, dt, \quad \forall\, x \in Z, \quad Px = Qx, \quad \forall\, x \in X.$$

This way, we have $\ker L = \mathrm{Im}P = \mathbb{R}$ and $\mathrm{coker}L = \mathrm{Im}Q = \mathbb{R}$. Moreover, we take J as the identity in $\mathbb{R}$. Notice that, for each $w \in Z$, the vector $v = K_P(I - Q)w$ is the (unique) solution of the linear boundary value problem

$$\begin{cases} -v'(t) = w(t) - \frac{1}{T} \int_0^T w(t)\, dt, \\ v(0) = v(T), \quad \int_0^T v(t)\, dt = 0. \end{cases}$$

Lastly, as nonlinear operator N, we take the associated Nemytskii operator, namely

$$(Nx)(t) := \psi(t, x(t)), \quad \forall\, x \in X.$$

By a standard argument, it is possible to verify that the operator N is L-completely continuous and, moreover, the map $\tilde{x}(\cdot)$ is a T-periodic solution of (1) if and only if $\tilde{x} \in \mathrm{dom}L$ with $L\tilde{x} = N\tilde{x}$. Analogously, solutions to the abstract equation $Lx = \lambda Nx$, with $0 < \lambda \leq 1$, correspond to T-periodic solutions of

$$x' + \lambda\psi(t,x) = 0, \quad 0 < \lambda \leq 1. \tag{2}$$

In the next two lemmas we provide some a priori bounds for the solutions of the parameter dependent equation (2) that will be useful for the application of Theorem 5.1 in the Appendix to the equation (1).

LEMMA 2.1. *Let $\psi : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying*

(H_0) $\exists\, a_0\,, b_0 \in L^1([0,T],\mathbb{R}^+)$: $\psi(t,x) \geq -a_0(t)|x| - b_0(t)$, $\forall\, x \in \mathbb{R}$ *and a.e.* $t \in [0,T]$.

Then, there exist constants $C \geq 1$ and $K > 0$ such that any T-periodic solution of (2) satisfies

$$\begin{cases} x_{\max} \leq C^{-1}x_{\min} + C^{-1}K & \textit{if } x_{\min} < -K, \\ |x(t)| \leq K, \, \forall\, t & \textit{if } -K \leq x_{\min} < 0, \\ x_{\max} \leq Cx_{\min} + K, & \textit{if } x_{\min} \geq 0. \end{cases} \tag{3}$$

Moreover, in any case

$$|x|_{\max} \leq C|x|_{\min} + K, \tag{4}$$

with $C = 1$ when $a_0 \equiv 0$.

Proof. Without loss of generality, let us suppose that $x_{\min} < x_{\max}$ and let $t_0 < t_1 < t_0 + T$ be such that $x(t_0) = x_{\min}$ and $x(t_1) = x_{\max}$. The theory of differential inequalities guarantees that, for all $t \in [t_0,t_1]$, we have that $x(t) \leq y(t)$, where y is the solution of the initial value problem

$$y' = a_0(t)|y| + b_0(t), \quad y(t_0) = x(t_0) = x_{\min}\,. \tag{5}$$

Notice that the solution $y(t)$ of the equation in (5) is monotone non-decreasing and therefore $y(t) \geq y(t_0)$ for all $t \in [t_0,t_1]$.

First of all, let us suppose that $x_{\min} = y(t_0) < 0$ and let $[t_0,\hat{t}[$ be the maximal open interval contained in $[t_0,t_1[$ such that $y(t) < 0$. Accordingly,

$$y'(t) = -a_0(t)y(t) + b_0(t), \quad \text{for a.e. } \ t \in [t_0,\hat{t}].$$

An integration of the linear equation on $[t_0,t] \subseteq [t_0,\hat{t}]$ yields to

$$\begin{aligned} y(t) &= y(t_0)\exp(-\mathcal{A}(t)) + \int_{t_0}^{t} b_0(\xi)\exp(\mathcal{A}(\xi) - \mathcal{A}(t))\,d\xi \\ &\leq y(t_0)\exp(-\mathcal{A}(t)) + \mathcal{B}(t), \end{aligned}$$

where we have set

$$\mathcal{A}(t) := \int_{t_0}^{t} a_0(\xi)\, d\xi, \quad \mathcal{B}(t) := \int_{t_0}^{t} b_0(\xi)\, d\xi.$$

Using the fact that $y(t_0) < 0$, it follows that

$$\begin{aligned} x(t) \le y(t) &\le \exp(-\mathcal{A}(t)) y(t_0) + \mathcal{B}(t) \\ &\le \exp\left(-\int_0^T a_0(t)\, dt\right) x_{\min} + \int_0^T b_0(t)\, dt \end{aligned}$$

holds for all $t \in [t_0, \hat{t}]$. By setting

$$K := \exp\left(\int_0^T a_0(t)\, dt\right) \int_0^T b_0(t)\, dt,$$

we immediately obtain that $y(t) < 0$ for all $t \in [t_0, \hat{t}]$ if $x_{\min} < -K$ and therefore, by the maximality of $\hat{t}$ we conclude that $\hat{t} = t_1$. Hence,

$$x_{\max} = x(t_1) \le y(t_1) \le \exp\left(-\int_0^T a_0(t)\, dt\right) x_{\min} + \int_0^T b_0(t)\, dt$$

and this proves the first inequality in (3) for

$$C := \exp\left(\int_0^T a_0(t)\, dt\right).$$

On the other hand, if $-K \le x_{\min} < 0$, either $x(t) \le 0$ for all $t \in [t_0, t_1]$, or $x_{\max} > 0$ and there exists a first time $\hat{t} \in [t_0, t_1[$ such that $x(\hat{t}) = 0$. By assumption, $-K \le x_{\min} \le x(t) \le 0$ for all $t \in [t_0, \hat{t}]$, while $x(t) \le v(t)$ on $[\hat{t}, t_1]$, where v is the solution of

$$v' = a_0(t) v + b_0(t), \quad v(\hat{t}) = x(\hat{t}) = 0\,.$$

An integration of the linear equation on $[\hat{t}, t_1]$ yields to

$$\begin{aligned} x_{\max} = x(t_1) \le v(t_1) &= \int_{\hat{t}}^{t_1} b_0(\xi) \exp(\mathcal{A}(t) - \mathcal{A}(\xi))\, d\xi \\ &\le \exp\left(\int_0^T a_0(t)\, dt\right) \int_0^T b_0(t)\, dt = K. \end{aligned}$$

Hence, in any case, we can conclude that $-K \le x_{\min} \le x(t) \le x_{\max} \le K$, for all t and the second inequality in (3) is verified.

At last, let us suppose that $x_{\min} = y(t_0) \geq 0$, so that (5) takes the form

$$y' = a_0(t)y + b_0(t), \quad \text{for a.e. } \ t \in [t_0, t_1].$$

An integration of the linear equation yields to

$$\begin{aligned} y(t) &= y(t_0)\exp(\mathcal{A}(t)) + \int_{t_0}^{t} b_0(\xi)\exp(\mathcal{A}(t) - \mathcal{A}(\xi))\,d\xi \\ &\leq (y(t_0) + \mathcal{B}(t))\exp(\mathcal{A}(t)) \leq (x_{\min} + \mathcal{B}(t))\exp(\mathcal{A}(t)). \end{aligned}$$

Therefore,

$$x_{\max} = x(t_1) \leq y(t_1) \leq Cx_{\min} + K$$

and the third inequality in (3) is verified.

Finally, (4) follows straightforwardly from (3) □

REMARK 2.2. It is crucial to observe that the constants C and K in Lemma 2.1 depend only on a_0 and b_0 and do not depend on the function ψ or the parameter $\lambda \in]0, 1]$.

For the main results of this section let us introduce the following definitions.

DEFINITION 2.3. *Let $\alpha \in W^{1,1}_T$. We say that α is a lower solution of* (1) *if*

$$\alpha'(t) + \psi(t, \alpha(t)) \leq 0, \quad \text{for a.e. } t \in [0, T]. \tag{6}$$

If α is not a solution, we say that it is proper. In particular, if

$$\alpha'(t) + \psi(t, \alpha(t)) < 0, \quad \text{for a.e. } t \in [0, T], \tag{7}$$

we say that the lower solution α is strongly proper.

An *upper solution* of (1) is defined in the same manner, just by reversing the inequality in (6) (respectively in (7), when it is strongly proper). Given $u, v \in C_T$, we denote by $u \leq v$ if $u(t) \leq v(t)$ for all $t \in [0, T]$ and by $u \prec v$ if $u \leq v$ and $u \not\equiv v$.

In the next definition we recall Villari's conditions [30] which is presented here in a slightly modified form. For other generalizations in different contexts, we refer to [4, 11, 21].

DEFINITION 2.4. *We say that $\psi(t, x)$ satisfies the Villari's condition at $-\infty$ (respectively, at $+\infty$) if, given $K_1 > 0$ and $K_2 > 0$, there exists a constant $d_0 > 0$ such that*

$$\exists\, \delta = \pm 1 : \ \delta \int_0^T \psi(t, x(t))\,dt > 0$$

for each $x \in C_T$ such that $x(t) \leq -d_0$, $\forall\, t \in [0, T]$ (respectively, $x(t) \geq d_0$, $\forall\, t \in [0, T]$) and $|x|_{\max} \leq K_1 |x|_{\min} + K_2$.

Now we are in position to state the following.

Theorem 2.5. *Let $\psi : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying (H_0) and the Villari's condition at $-\infty$ with $\delta = 1$. Suppose there exists $\alpha \in W^{1,1}_T$ which is a strongly proper lower solution for equation (1). Then, (1) has at least a T-periodic solution $\tilde{x}$ such that $\tilde{x} \prec \alpha$. Moreover, there exists $R_0 \geq d_0$ such that any T-periodic solution of (1) with $x \leq \alpha$, satisfies $x(t) > -R_0$ for all $t \in [0,T]$.*

Let us make a comment before proceeding with the proof of the theorem. In presence of a lower solution, one can expect to find a solution $\tilde{x} \geq \alpha$. Indeed, what we are going to do, is to treat α as an upper solution of the problem. Our notation is consistent with the one in [7, 25], nevertheless other authors overturn the terminology (cf. [24]). Actually, for Theorem 2.5 the terminology is not relevant and what matters is that α satisfies (7).

Proof. Following a standard approach, we define the truncated function

$$\hat{\psi}(t,x) := \begin{cases} \psi(t,x) & \text{for } x \leq \alpha(t), \\ \psi(t,\alpha(t)) & \text{for } x \geq \alpha(t), \end{cases}$$

and consider the parameter dependent equation

$$x' + \lambda\hat{\psi}(t,x) = 0, \quad 0 < \lambda \leq 1. \tag{8}$$

First of all, as a consequence of (H_0), we remark that

$$\hat{\psi}(t,x) \geq -a_0(t)|x| - b_1(t),\ \forall\, x \in \mathbb{R} \text{ and a.e. } t \in [0,T],$$

where $b_1(t) = b_0(t) + a_0(t)|\alpha(t)|$. Therefore $\hat{\psi}$ satisfies (H_0), too. According to Lemma 2.1 (applied to $\hat{\psi}$ in place of ψ) any T-periodic solution x of (8) satisfies

$$|x|_{\max} \leq K_1|x|_{\min} + K_2$$

for some suitable constants $K_1 \geq 1$ and $K_2 > 0$ possibly depending in α but independent on x and λ.

Next, we choose a constant $d_1 \geq d_0$ with $d_1 > \|\alpha\|_\infty$ and we claim that $\max x > -d_1$. Indeed, if we suppose by contradiction that $x(t) \leq -d_1$ for all $t \in [0,T]$, then $x(t) < \alpha(t)$ for all $t \in [0,T]$ and so $x(t)$ is a T-periodic solution of (2). Hence, an integration on $[0,T]$ of (2) (divided by $\lambda > 0$), yields to $\int_0^T \psi(t,x(t))\,dt = 0$, which clearly contradicts Villari's condition at $-\infty$ as $-d_1 \leq -d_0$. Having proved that $x(t) > -d_1$ for some $t \in [0,T]$ and hence $\max x > -d_1$, we obtain that

$$\min x > -R_0\,, \quad \text{for } R_0 := K_1 d_1 + K_2\,.$$

Now, we claim that there exists $\bar{t} \in [0,T]$ such that $x(\bar{t}) < \alpha(\bar{t})$. If, by contradiction, $x(t) \geq \alpha(t)$ for all $t \in [0,T]$, then x is a T-periodic solution of $x' + \lambda\psi(t,\alpha(t)) = 0$, for $0 < \lambda \leq 1$ and then an integration on $[0,T]$ of this equation (divided by $\lambda > 0$), yields to $\int_0^T \psi(t,\alpha(t))\,dt = 0$. On the other hand, an integration of (7) on $[0,T]$ gives $\int_0^T \psi(t,\alpha(t))\,dt < 0$, thus a contradiction. Having proved that $x(t) < \|\alpha\|_\infty$ for some $t \in [0,T]$ and hence $\min x < \|\alpha\|_\infty$, we can also deduce that

$$\max x < K_1\|\alpha\|_\infty + K_2\,.$$

Writing equation

$$-x' = \hat{\psi}(t,x) \tag{9}$$

as a coincidence equation of the form $Lx = \hat{N}x$ in the space C_T, from the a priori bounds, we find that the coincidence degree $D_L(L-\hat{N},\mathcal{O})$ is well defined for any open and bounded set $\mathcal{O} \subset C_T$ of the form

$$\mathcal{O} := \{x \in C_T : -R^- < x(t) < R^+,\ \forall\, t \in [0,T]\}$$

where $R^- \geq R_0$, $R^+ \geq K_1\|\alpha\|_\infty + K_2$.

As a last step, we consider the averaged scalar map

$$\hat{\psi}^\# : \mathbb{R} \to \mathbb{R}, \quad \hat{\psi}^\#(\xi) := \frac{1}{T}\int_0^T \hat{\psi}(t,\xi)\,dt, \quad \forall\, \xi \in \mathbb{R}.$$

We have $-JQ\hat{N}|_{\ker L} = -\hat{\psi}^\#$ and $\hat{\psi}^\#(-R^-) > 0 > \hat{\psi}^\#(R^+)$.
In more detail, since $R^- \geq d_1$, the first inequality follows from Villari's condition, while $\int_0^T \psi(t,\alpha(t))\,dt < 0$ and the choice $R^+ \geq \|\alpha\|_\infty$, imply the second inequality. An application of Theorem 5.1 guarantees that $D_L(L-\hat{N},\mathcal{O}) = 1$ and hence equation (9) has a T-periodic solution $\tilde{x}$ with $-R^- < \tilde{x}(t) < R^+$, for all $t \in [0,T]$.

In order to conclude, we check that $\tilde{x} \prec \alpha$. This is a standard fact, however we give the details for the reader's convenience. From the previous part of the proof we already know that any T-periodic solution of (8) is below α, at least for some t, thus the same must occur for $\tilde{x}$. Let t_* be such that $\tilde{x}(t_*) < \alpha(t_*)$. Suppose, by contradiction, that there exists a t^* such that $\tilde{x}(t^*) > \alpha(t^*)$. Let $[t_1,t_2]$ be such that $t_1 < t^* < t_2$ with $v(t) > 0$ for all $t \in]t_1,t_2[$ and, moreover, $v(t_1) = v(t_2) = 0$. On the interval $[t_1,t_2]$, we have that $\tilde{x}'(t)+\psi(t,\alpha(t)) = 0$ and hence, recalling (7), we find that $v'(t) > 0$, for a.e. $t \in [t_1,t_2]$. An integration on $[t_1,t_2]$ gives immediately a contradiction. We have thus proved that $\tilde{x}(t) \leq \alpha(t)$ for all $t \in [0,T]$ and therefore $\tilde{x}$ is a T-periodic solution of (1) satisfying $\tilde{x} \leq \alpha$. Moreover, since α is proper, we conclude that $\tilde{x} \prec \alpha$. $\square$

Remark 2.6. Notice that, under additional hypothesis ensuring that the T-periodic solutions x with $x \leq \alpha$ are such that $x \ll \alpha$, namely $x(t) < \alpha(t)$ for all t, we can also prove that:

> there exist $R_0 \geq d_0$ such that for each $R > R_0$, we have $D_L(L-N,\Omega) = 1$ for $\Omega = \{x \in C_T : -R < x(t) < \alpha(t)\,\forall\, t \in [0,T]\}$.

A possible additional hypothesis guaranteeing $x \ll \alpha$ could be

(A) For all $t_0 \in [0,T]$ and $u_0 \in \mathbb{R}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $|t-t_0| < \delta$, $|u-u_0| < \delta \Rightarrow |\psi(t,u) - \psi(t,u_0)| < \varepsilon$.

Observe that (A) is always satisfied when ψ is continuous. Such kind of conditions are widely discussed in [6] for second order equations.

We propose now a dual version Theorem 2.5 whose proof can be obtained via minor changes.

Theorem 2.7. *Let $\psi : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying (H_0) and the Villari's condition at $+\infty$ with $\delta = 1$. Suppose there exists $\alpha \in W_T^{1,1}$ which is a strongly proper lower solution for equation* (1). *Then,* (1) *has at least a T-periodic solution $\tilde{x}$ such that $\tilde{x} \succ \alpha$. Moreover, there exists $R_0 \geq d_0$ such that any T-periodic solution of* (1) *with $x \geq \alpha$, satisfies $x(t) < R_0$ for all $t \in [0,T]$.*

Proof. We define the truncated function

$$\hat{\psi}(t,x) := \begin{cases} \psi(t,x) & \text{for } x \geq \alpha(t), \\ \psi(t,\alpha(t)) & \text{for } x \leq \alpha(t), \end{cases}$$

and consider the parameter dependent equation (8). The proof now follows the same scheme as that of Theorem 2.5 till to the introduction of an open bounded set $\mathcal{O} := \{x \in C_T : -S^- < x(t) < S^+,\ \forall\, t \in [0,T]\}$ where S^- and S^+ are suitable constants obtained similarly as R^- and R^+. In this case, one can compute the coincidence degree and find that $D_L(L-\hat{N},\mathcal{O}) = -1$, thus ensuring the existence of a T-periodic solution $\tilde{x} \in \mathcal{O}$. Finally, by the same argument as above, we prove that $\tilde{x} \succ \alpha$. □

It is a well-known fact (cf. [14]), that results like Theorem 2.5 or Theorem 2.7, obtained by using strict inequalities, can be relaxed by considering weak inequalities. Accordingly, from Theorem 2.5, the following result holds.

Corollary 2.8. *Let $\psi : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying (H_0) and such that, given $K_1 > 0$ and $K_2 > 0$, there exists $d_0 > 0$ for which $\int_0^T \psi(t,x(t))\,dt \geq 0$ for each $x \in C_T$ with $x(t) \leq -d_0$, $\forall\, t \in [0,T]$ and $|x|_{\max} \leq K_1|x|_{\min} + K_2$. Suppose there exists a lower solution $\alpha \in W_T^{1,1}$ for equation* (1). *Then,* (1) *has at least a T-periodic solution $\tilde{x}$ such that $\tilde{x} \leq \alpha$.*

Proof. We introduce the auxiliary functions

$$\ell(x) := \max\{-1, -x - \|\alpha\|_\infty - 1\}, \quad \psi^\varepsilon(t,x) := \psi(t,x) + \varepsilon\ell(x),\ \varepsilon > 0$$

and apply Theorem 2.5 to equation $x' + \psi^\varepsilon(t,x) = 0$. Moreover, one can easily check that the constant R_0 can be taken uniformly with respect to ε. The conclusion then follows via Ascoli-Arzelà theorem. □

A corollary similar to the above one can be stated with respect to Theorem 2.7.

3. Existence and multiplicity theorems

Here we discuss the number of T-periodic solutions for the parameter dependent equation

$$(\mathscr{E}_s) \qquad x' + g(t,x) = s.$$

Throughout this section we suppose that $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions.

Moreover, in the sequel, the following hypotheses will be considered:

(G_0) $\exists\, a_0, b_0 \in L^1([0,T], \mathbb{R}^+) :\ g(t,x) \geq -a_0(t)|x| - b_0(t),\ \forall\, x \in \mathbb{R}$ and a.e. $t \in [0,T]$;

(G_1) $\exists\, x_0\,, g_0 \in \mathbb{R} :\ g(t,x_0) \leq g_0$ for a.e. $t \in [0,T]$;

(G_2^-) given $K_1 > 0$ and $K_2 > 0$, for each σ there exists $d_\sigma > 0$ such that $\frac{1}{T}\int_0^T g(t,x(t))\,dt > \sigma$ for each $x \in C_T$ such that $x(t) \leq -d_\sigma$ for all $t \in [0,T]$ and $|x|_{\max} \leq K_1|x|_{\min} + K_2$;

(G_2^+) given $K_1 > 0$ and $K_2 > 0$, for each σ there exists $d_\sigma > 0$ such that $\frac{1}{T}\int_0^T g(t,x(t))\,dt > \sigma$ for each $x \in C_T$ such that $x(t) \geq d_\sigma$ for all $t \in [0,T]$ and $|x|_{\max} \leq K_1|x|_{\min} + K_2$.

THEOREM 3.1. *Assume (G_0), (G_1) and, either (G_2^-) or (G_2^+). Then, there exists $s_0 \in \mathbb{R} \cup \{-\infty\}$ such that for every $s > s_0$ equation $(\mathscr{E}_s)$ has at least one T-periodic solution.*

Proof. For any given parameter $s \in \mathbb{R}$, we set

$$\psi_s(t,x) := g(t,x) - s,$$

so that equation $(\mathscr{E}_s)$ is of the form (1). Just to fix a case, let us suppose that (G_2^-) holds.

We start by choosing a parameter $s_1 > g_0$. In this situation, the constant function $\alpha(t) \equiv x_0$ is a strongly proper lower solution. Indeed, we have

$$\alpha'(t) + g(t, \alpha(t)) - s_1 = g(t, x_0) - s_1 \leq -(s_1 - g_0) < 0$$

On the other hand, for $\sigma = s_1$, condition (G_2^-) implies the Villari's condition at $-\infty$ with $\delta = 1$. Hence, an application of Theorem 2.5 guarantees the existence of at least one T-periodic solution x of $(\mathscr{E}_{s_1})$ with $x \prec x_0$.

Next, we claim that if, for some $\tilde{s} < s_1$ the equation has a T-periodic solution (that we will denote by w), then equation $(\mathscr{E}_s)$ has a T-periodic solution for each $s \in]\tilde{s}, s_1[$. We write equation $(\mathscr{E}_s)$ as

$$x' + g(t, x) - \tilde{s} - (s - \tilde{s}) = 0,$$

so that $\alpha(t) \equiv w(t)$ is a strongly proper lower solution of $(\mathscr{E}_s)$. Indeed, we have

$$\alpha'(t) + g(t, \alpha(t)) - s = w'(t) + g(t, w(t)) - s = -(s - \tilde{s}) < 0.$$

On the other hand, for $\sigma = s$, condition (G_2^-) implies the Villari's condition at $-\infty$ with $\delta = 1$. An application of Theorem 2.5 guarantees the existence of at least one T-periodic solution x of $(\mathscr{E}_s)$ with $x \prec w$ and the claim is proved.

Since we can take s_1 arbitrarily large, we conclude that the set of the parameters s for which equation $(\mathscr{E}_s)$ has T-periodic solutions is an interval $\mathcal{J}$ with $\sup \mathcal{J} = +\infty$. Setting

$$s_0 := \inf\{s \in \mathbb{R} : (\mathscr{E}_s) \text{ has at least one } T\text{-periodic solution}\} \in \mathbb{R} \cup \{-\infty\},$$

the thesis follows. The same argument applies if, instead of (G_2^-), we assume (G_2^+) and apply Theorem 2.7. $\square$

Remark 3.2. Let us make some comments that arise from Theorem 3.1. The first one is about the critical parameter s_0. Without supplementary conditions, we cannot say, a priori, whether $s_0 = -\infty$ or $s_0 \in \mathbb{R}$ and, in this latter case, if the equation $(\mathscr{E}_{s_0})$ has T-periodic solutions. Simple examples can be provided for each of these cases. However, from the proof, it is clear that $s_0 \leq g_0$. As a second comment, we observe that the Villari's conditions $(G_2^{\pm})$ guarantee the existence of upper solutions. In fact, suppose that w is a T-periodic solution of $(\mathscr{E}_{s_1})$ for some $s_1 > g_0$. Then $\beta(t) \equiv w(t)$ is a strongly proper upper solution of $(\mathscr{E}_s)$ for any $s < s_1$. Indeed, we have $\beta'(t) + g(t, \beta(t)) - s = w'(t) + g(t, w(t)) - s = s_1 - s > 0$. Hence, a posteriori along the proof, we have discovered that for $s \in]g_0, s_1[$, there are both a strongly proper upper solution β and a strongly proper lower solution α with $\beta \prec \alpha$ or $\alpha \prec \beta$, according to the assumption (G_2^-) or (G_2^+), respectively. Thus we enter in the setting of [25] where a detailed analysis is performed about continua of T-periodic solutions and their stability.

The previous result concerns the case in which the conditions $(G_2^{\pm})$ are applied in a separately way. The next theorem considers the situations in which Villari's conditions hold at the same time.

THEOREM 3.3. *Assume* (G_0), (G_1), (G_2^-) *and* (G_2^+)*. Then there exists* $s_0 \in \mathbb{R}$ *such that:*

1° *for* $s < s_0$, *equation* $(\mathscr{E}_s)$ *has no* T*-periodic solutions;*

2° *for* $s = s_0$, *equation* $(\mathscr{E}_s)$ *has at least one* T*-periodic solution;*

3° *for* $s > s_0$, *equation* $(\mathscr{E}_s)$ *has at least two* T*-periodic solutions.*

Proof. Without loss of generality, we can suppose that the map $\sigma \mapsto d_\sigma$ is defined on $[0,+\infty)$ and is monotone non-decreasing.

We claim that there exists a constant $\nu_0 \leq 0$ such that, if $s < \nu_0$, equation $(\mathscr{E}_s)$ has no T-periodic solution.

Indeed, let x be a T-periodic solution of $(\mathscr{E}_s)$ for any $s \leq 0$. The function $\psi_s(t,x) = g(t,x) - s$ satisfies condition (H_0), uniformly for $s \leq 0$. Hence, according to Lemma 2.1 and Remark 2.2, there exist two constants $C \geq 1$ and $K > 0$ such that (4) holds for each T-periodic solution of $(\mathscr{E}_s)$. Consider now condition (G_2^+) that we read now for $\sigma = 0$ and $K_1 = C$, $K_2 = K$. It implies that if $x(t) \geq d_0$ for all $t \in [0,T]$, then $\int_0^T g(t,x(t))\,dt > 0$. On the other hand, $x' + g(t,x) = s \leq 0$ and a contradiction follows. This implies that $x_{\min} < d_0$. In the same manner, using (G_2^-) for $\sigma = 0$ and $K_1 = C$, $K_2 = K$, we can prove that $x_{\max} > -d_0$. In conclusion, we have proved that $|x|_{\min} < d_0$. Therefore, from (4) we find that

$$|x|_{\max} < R^* := Cd_0 + K. \tag{10}$$

We stress the fact that (10) holds for any possible Tperiodic solution of $(\mathscr{E}_s)$ with $s \leq 0$. Now, let $\rho \in L^1([0,T],\mathbb{R}^+)$ be such that

$$|g(t,\xi)| \leq \rho(t), \quad \forall\, \xi \in [-R^*,R^*] \quad \text{and a.e.} \quad t \in [0,T].$$

Let us consider again $x' + g(t,x) = s$ with $s \leq 0$. Integrating the equation on $[0,T]$, we have

$$sT = \int_0^T g(t,x(t))\,dt \geq -\|\rho\|_1\,.$$

We have thus proved that if there exists a T-periodic solution of $(\mathscr{E}_s)$ for $s \leq 0$, then, necessarily

$$s \geq \nu_0 := -\frac{1}{T}\|\rho\|_1\,.$$

Hence, if $s < \nu_0$, equation $(\mathscr{E}_s)$ has no T-periodic solution. The claim is proved.

After this preliminary observation, we proceed now as in the proof of Theorem 3.1. We fix (arbitrarily) $s_1 > g_0$ and using (G_2^-), as well as (G_2^+), we prove the existence of at least two T-periodic solutions $x^{(-)}$ and $x^{(+)}$ with $x^{(-)} \prec x_0 \prec x^{(+)}$.

Next, we claim that if, for some $\tilde{s} < s_1$ the equation has a T-periodic solution (that we will denote by w), then equation $(\mathscr{E}_s)$ has at least two T-periodic solutions for each $s \in]\tilde{s}, s_1[$.

We write equation $(\mathscr{E}_s)$ as

$$x' + g(t,x) - \tilde{s} - (s - \tilde{s}) = 0,$$

so that $\alpha(t) \equiv w(t)$ is a strongly proper lower solution of $(\mathscr{E}_s)$ (as in Theorem 3.1). On the other hand, for $\sigma = s$, condition (G_2^-) implies the Villari's condition at $-\infty$ with $\delta = 1$ and, similarly, (G_2^+) implies the Villari's condition at $+\infty$ with $\delta = 1$ An application of Theorem 2.5 and Theorem 2.7 guarantees the existence of at least one T-periodic solution $u^{(-)}$ of $(\mathscr{E}_s)$ with $u^{(-)} \prec w$ and the existence of at least one T-periodic solution $u^{(+)}$ of $(\mathscr{E}_s)$ with $u^{(+)} \succ w$. Clearly, $u^{(-)} \not\equiv u^{(+)}$.

Since we can take s_1 arbitrarily large, we conclude that the set of the parameters s for which equation $(\mathscr{E}_s)$ has T-periodic solutions is an interval $\mathcal{J}$ with $\sup \mathcal{J} = +\infty$. Setting

$$s_0 := \inf\{s \in \mathbb{R} : (\mathscr{E}_s) \text{ has at least one } T\text{-periodic solution}\} \in \mathbb{R} \cup \{-\infty\},$$

we know that s_0 is finite, indeed, $\nu_0 \leq s_0 \leq g_0$. Moreover, by the above discussion, we also know, that for each $s > s_0$ equation $(\mathscr{E}_s)$ has at least two T-periodic solutions. By construction, we also know that for $s < s_0$, there is no T-periodic solution for $(\mathscr{E}_s)$.

To conclude the proof, we have to check that for $s = s_0$ there is at least one T-periodic solution. This will be achieved following an argument borrowed from [8]. Let $s_2 < s_0 < s_1$ be fixed and let θ_n be a decreasing sequence of parameters with $\theta_n \to s_0$ and $\theta_n \in]s_0, s_1]$ for all n. By the estimates developed previously, we know that, for each n there exists at least one (actually two) T-periodic solution w_n of equation $x' + g(t,x) = \theta_n$ with $\|w_n\|_\infty \leq M$, where M is a uniform a priori bound obtained as R^* in (10). An application of the Ascoli-Arzelà theorem, passing to the limit as $n \to \infty$, provides the existence of at least one T-periodic solution of $(\mathscr{E}_s)$ for $s = s_0$. This completes the proof. $\square$

Remark 3.4. Notice that assuming the Villari's condition (GV) is equivalent to require both (G_2^-) and (G_2^+). As in [17, Remark 2], we also observe that all the results remain true if s in $(\mathscr{E}_s)$ is replaced by $s\varphi(t)$ with $\varphi \in L^\infty(0,T)$ and positive (i.e. $\operatorname{essinf}\varphi > 0$).

4. Applications

In this section we show a few applications of the preceding theorems in order to treat some classical examples in literature. In particular, we focus our attention to consequences of Theorem 3.3.

As a first example, we consider the periodic problem associated with

$$x' + \gamma(t)\phi(x) = s + p(t). \tag{$\mathscr{WE}_s$}$$

In this case, a multiplicity result reads as follow.

Corollary 4.1. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous function and suppose that*

$$\lim_{|x|\to\infty} \phi(x) = +\infty. \tag{H_ϕ}$$

Let γ, $p \in L^\infty(0,T)$ with $\gamma(t) \geq 0$ for a.e. $t \in [0,T]$ and $\int_0^T \gamma(t)\,dt > 0$. Then, there exists $s_0 \in \mathbb{R}$ such that:

1° *for $s < s_0$, equation ($\mathscr{WE}_s$) has no T-periodic solutions;*

2° *for $s = s_0$, equation ($\mathscr{WE}_s$) has at least one T-periodic solution;*

3° *for $s > s_0$, equation ($\mathscr{WE}_s$) has at least two T-periodic solutions.*

Proof. We apply Theorem 3.3 for

$$g(t,x) := \gamma(t)\phi(x) - p(t).$$

Let us set $\phi_0 := \min_{\xi\in\mathbb{R}} \phi(\xi)$. For any $d > \max\{\phi_0, 0\}$, we introduce the following constants:

$$\zeta^-(d) := \min\{\phi(x) : x \leq -d\}, \quad \zeta^+(d) := \min\{\phi(x) : x \geq d\}.$$

From (H_ϕ) it follows that $\zeta^\pm(d) \to +\infty$ for $d \to +\infty$.

Let $x \in C_T$ be such that $|x(t)| \geq d > 0$ for all $t \in [0,T]$. If $x(t) \leq -d$, $\forall\, t$, then

$$\begin{aligned}\frac{1}{T}\int_0^T g(t,x(t))\,dt &= \frac{1}{T}\int_0^T \gamma(t)\phi(x(t))\,dt - \frac{1}{T}\int_0^T p(t)\,dt\\ &\geq \frac{\zeta^-(d)}{T}\int_0^T \gamma(t)\,dt - \frac{1}{T}\int_0^T p(t)\,dt.\end{aligned}$$

In the other case, if $x(t) \geq d$, $\forall\, t$, then

$$\frac{1}{T}\int_0^T g(t,x(t))\,dt \geq \frac{\zeta^+(d)}{T}\int_0^T \gamma(t)\,dt - \frac{1}{T}\int_0^T p(t)\,dt.$$

Hence, the Villari's condition (GV) is satisfied by the properties of $\zeta^{\pm}(d)$.

Hypothesis (G_0) is satisfied by choosing as $b_0(t)$ the positive part of $p(t) - \gamma(t)\phi_0$ and $a_0 \equiv 0$. Also (G_1) holds for $x_0 = 0$ and any constant $g_0 \geq \|\gamma\|_\infty \phi(0) + \|p\|_\infty$. Now, an application of Theorem 3.3 gives the result. □

Corollary 4.1 extends [24, Corollary 3.1], where the periodic problem for equation $(\mathscr{WE}_s)$ was considered only for $\gamma \equiv 1$.

Our second example deals with a generalized Riccati equation of the form

$$x' + \gamma_p(t)|x|^p + \gamma_1(t)x + \gamma_0(t) = s. \tag{$\mathscr{RE}_s$}$$

Also in this case a multiplicity result can be stated.

Corollary 4.2. *Let $\gamma_0 \in L^\infty(0,T)$ and $\gamma_1, \gamma_p \in L^1(0,T)$, with $\gamma_p(t) \geq 0$ for a.e. $t \in [0,T]$ and $\int_0^T \gamma_p(t)\,dt > 0$. Then, there exists $s_0 \in \mathbb{R}$ such that:*

1° *for $s < s_0$, equation $(\mathscr{RE}_s)$ has no T-periodic solutions;*

2° *for $s = s_0$, equation $(\mathscr{RE}_s)$ has at least one T-periodic solution;*

3° *for $s > s_0$, equation $(\mathscr{RE}_s)$ has at least two T-periodic solutions.*

Proof. We show, that all the hypotheses of Theorem 3.3 are fulfilled for

$$g(t,x) := \gamma_p(t)|x|^p + \gamma_1(t)x + \gamma_0(t).$$

Condition (G_0) holds for $a_0 := |\gamma_1|$ and $b_0 := |\gamma_0|$. Concerning hypothesis (G_1) we observe that it is satisfied with $x_0 = 0$ and $g_0 \geq \|\gamma_0\|_\infty$. Finally, we verify the validity of the Villari's condition (GV). Let us suppose that $K_1 \geq 1$ and $K_2 > 0$ are fixed and $x \in C_T$ is such that $|x|_{\max} \leq K_1|x|_{\min} + K_2$.

$$\begin{aligned} \frac{1}{T}\int_0^T g(t,x(t))\,dt &= \frac{1}{T}\int_0^T \big(\gamma_p(t)|x(t)|^p + \gamma_1(t)x(t) + \gamma_0(t)\big)\,dt \\ &\geq |x|_{\min}^p \|\gamma_p\|_1 - |x|_{\max}\|\gamma_1\|_1 - \|\gamma_0\|_1 \\ &\geq |x|_{\min}^p \|\gamma_p\|_1 - |x|_{\min}K_1\|\gamma_1\|_1 - K_2\|\gamma_1\|_1 - \|\gamma_0\|_1 \,. \end{aligned}$$

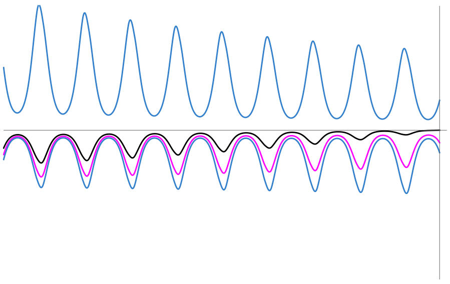
Therefore,

$$\frac{1}{T}\int_0^T g(t,x(t))\,dt \to +\infty, \quad \text{as } |x|_{\min} \to +\infty,$$

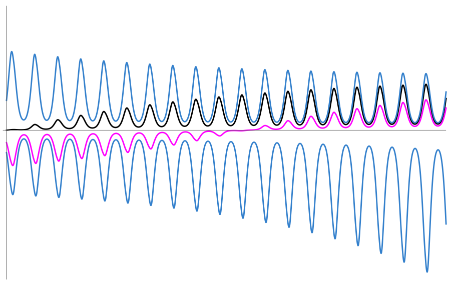
so that (GV) is satisfied. □

Remark 4.3. The nonlinear term $\gamma_p(t)|x|^p + \gamma_1(t)x + \gamma_0(t)$ in equation $(\mathscr{RE}_s)$ is convex in x (and strictly on a set of positive measure). We can then apply a result of Mawhin in [17, Proposition 3] which guarantees that there are at most two T-periodic solutions for each $s \in \mathbb{R}$. As a consequence, in the situation of

Corollary 4.2, we conclude that for each $s > s_0$ equation $(\mathscr{RE}_s)$ has exactly two T-periodic solutions $x^{(-)} < x^{(+)}$. Moreover, $x^{(+)}$ is asymptotically stable and $x^{(-)}$ is unstable (cf. [25]). Figure 1 shows an example for this case. The same conclusion holds also for Corollary 4.1 if we assume that ϕ is strictly convex.



(a) The four solutions in the interval $[-60, 0]$ show evidence of the presence of an unstable periodic solution.



(b) The four solutions in the interval $[0, 120]$ show evidence of an asymptotically stable periodic solution.

Figure 1: A numerical simulation for equation $(\mathscr{RE}_s)$. The example is obtained for $\gamma_2(t) = \max\{0, \sin t - 0.9\}$, $\gamma_1(t) = \cos t$, $\gamma_0(t) = 0$, $p = 1.1$ and $s = 1$. We have considered the solutions corresponding to four initial points $x(0) = -90, -50\,(\text{magenta}), 0\,(\text{black}), 120$. Consistently with Remark 4.3 we give evidence of two 2π-periodic solutions.

5. Appendix: Mawhin's coincidence degree

For the reader's convenience, we briefly recall here a few basic facts from coincidence degree theory which are used in the present paper. We refer to [10, 15, 19] for the general theory.

Let X, Z be real normed spaces and let Ω be an open bounded set in X. We consider a coincidence equation of the form

$$Lx = Nx, \quad x \in \mathrm{dom}L \cap \Omega, \tag{11}$$

where $L : X \supseteq \mathrm{dom}L \to Z$ is a linear (non-invertible) Fredholm mapping of index zero and $N : X \to Z$ is a nonlinear operator. We also consider two linear and continuous projections $P : X \to \ker L$ and $Q : Z \to \mathrm{Im}L$, as well as, the (continuous) right inverse of L, denoted by $K_P : \mathrm{Im}L \to \mathrm{dom}L \cap X_0$, where $X_0 := \ker P \equiv X/\ker L$ is a complementary subspace of $\ker L$ in X. In this manner (11) is equivalent to the fixed point problem

$$x = \Phi(x) := Px + JQNx + K_P(I - Q)Nx, \quad x \in \Omega, \tag{12}$$

where $J : \text{coker}L = \text{Im}Q \equiv Z/\text{Im}L \to \ker L$ is a linear isomorphism. We further suppose that N is a continuous operator which maps bounded sets to bounded sets and such that, for any bounded set B in X, the set $K_P(I-Q)N(B)$ is relatively compact (i.e., N is L-completely continuous [19]). As a consequence, the operator Φ, defined in (12), is completely continuous, too.

If we suppose that

$$Lx \neq Nx, \quad \forall\, x \in \text{dom}L \cap \partial\Omega,$$

then also $I - \Phi$ never vanishes on $\partial\Omega$ and, therefore, we can define the *coincidence degree*

$$D_L(L-N,\Omega) := \deg(I-\Phi,\Omega,0),$$

where "deg" denotes the Leray-Schauder degree. Notice that, usually one defines the coincidence degree with absolute value, namely $|D_L(L-N,\Omega)| = |\deg(I-\Phi,\Omega,0)|$ in order to make the degree independent from the choice of the projections P, Q, the isomorphism J and the orientations of $\ker L$ and $\text{coker}L$ (see [19]). In our applications no sign ambiguity will arise because we fix the natural orientations on $\ker L$ and $\text{coker}L$, which are identified by $\mathbb{R}$ and we choose P, Q and J in an obvious way.

If we denote by "$\deg_B$" the (finite dimensional) Brouwer degree, then, according to Mawhin's continuation theorem (see [12, 13]), the following result holds.

Theorem 5.1. *Let L and N be as above and let $\Omega \subseteq X$ be an open and bounded set. Suppose that $Lx \neq \lambda Nx$, $\forall\, x \in \text{dom}\, L \cap \partial\Omega$, $\forall\, \lambda \in\,]0,1]$ and $QN(x) \neq 0$, $\forall\, x \in \partial\Omega \cap \ker L$. Then,*

$$D_L(L-N,\Omega) = \deg_B(-JQN|_{\ker L}, \Omega \cap \ker L, 0).$$

As a consequence, if $\deg_B(-JQN|_{\ker L}, \Omega \cap \ker L, 0) \neq 0$, then (11) has at leat one solution.

We also point out that the classical properties of the Leray-Schauder degree, such as additivity/excision, homotopic invariance, hold also in the coincidence degree framework.

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Received October 23, 2017
Accepted October 27, 2017